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Quantization with maximally degenerate Poisson brackets: the harmonic oscillator!

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Abstract

Nambu's construction of multi-linear brackets for super-integrable systems can be thought of as degenerate Poisson brackets with a maximal set of Casimirs in their kernel. By introducing privileged coordinates in phase space these degenerate Poisson brackets are brought to the form of Heisenberg's equations. We propose a definition for constructing quantum operators for classical functions, which enables us to turn the maximally degenerate Poisson brackets into operators. They pose a set of eigenvalue problems for a new state vector. The requirement of the single-valuedness of this eigenfunction leads to quantization. The example of the harmonic oscillator is used to illustrate this general procedure for quantizing a class of maximally super-integrable systems.

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1. Introduction

The passage from classical to quantum mechanics is based on the Hamiltonian formulation of the classical system. This is most apt because one of the crowning achievements of Newtonian mechanics was the construction of its underlying theory of symplectic structure. The great works of Hamilton, Poisson, Jacobi and Darboux in the late 19th century gave it the appearance of a finished gem. However, it was only in the second half of the 20th century that we came to understand this theory to be much richer and certainly very far from complete [1]. Nowhere is this more manifest than in the theory of completely integrable systems where we encounter more than one Poisson structure as a matter of routine [2]. How are the modern developments in classical theory of Poisson structure going to reflect on the problem of quantization?

Nambu [3] published an alternative Hamiltonian formulation of super-integrable systems based on ideas he had as a student, but only when he was already one of the leading theoretical physicists of all time. The literature on the Nambu bracket has followed Nambu's idea of regarding it as a multi-linear object rather than bilinear as in the case of Poisson brackets. The problem of quantization with Nambu brackets has been discussed in the framework of deformation quantization [4–14]. In particular, we refer to the recent work of Curtright and Zachos [13] which serves as a textbook for this line of research. There is also a geometrical approach using complex projective Hilbert space endowed with Kähler structure [15, 16]. In this paper we shall adopt a new approach, completely different from these.

We shall avoid multi-linear products altogether and instead focus on taking Nambu brackets as Poisson brackets, albeit degenerate ones because they admit a full set of Casimirs in their kernel. The free Euler top that Nambu discussed as an illustration of his ideas is a classic example of super-integrable systems that arise from the existence of hidden symmetries. Super-integrability is much stronger than Liouville integrability because such systems are required to admit first integrals that number not half, but only one less than the dimension of the dynamical system itself. The discussion of the dynamics is then reduced to a single torus. Each one of these conserved quantities can alternatively be taken as the Hamiltonian function and each such choice results in the construction of an independent degenerate Poisson bracket. In this way we obtain the maximal number of compatible Poisson structures for super-integrable systems. This new perspective whereby the ideas of Nambu find realization as Poisson brackets was discussed only recently [17] (see also [21, 22]). To distinguish our approach from earlier literature it would have been natural to use the name Nambu–Poisson bracket, but this has been used before in another context and therefore we shall simply call it maximally degenerate Poisson brackets.

Poisson tensors for super-integrable systems that are maximally degenerate involve highly nonlinear expressions of the dynamical variables and we cannot naively carry them over to the quantum-mechanical domain. However, quantization with maximally degenerate Poisson brackets becomes possible if we adopt the strategy of introducing privileged coordinates in phase space and a new definition for the quantum operator corresponding to any given function of classical variables. In this way we construct quantum operators for each maximally degenerate Poisson bracket and the commutation relations are brought to the form of Heisenberg's equations. Thus, we set up a set of eigenvalue problems for a new state vector. The quantization condition is the single-valuedness of this eigenfunction which is a new object, not immediately related to Schrödinger's wavefunction.

We shall not present a formalism but instead give an outline of our approach interspersed with an illustrative example which is the simplest super-integrable system, namely the harmonic oscillator in two dimensions! This will help to fix ideas. Even for this simplest system the calculations are rather involved. Later on we shall consider the general formalism and other familiar problems which admit maximally degenerate Poisson structure, such as the Kepler problem [7], the example of S^2 that Curtright and Zachos [11] have discussed and Calogero-type systems [17] as well as the completely integrable Smorodinsky–Winternitz potentials [18, 19] of the Schrödinger equation [20].

2. Classical maximally degenerate Poisson brackets

We start with a $2n$-dimensional Hamiltonian system with the equations of motion given by

$$\dot{X}^A = [X^A, H^1]_0 = J_0^{AB} \frac{\partial H^1}{\partial X^B} \tag{1}$$

where $X^A \in \{q^1, \ldots, q^n, p_1, \ldots, p_n\}$ and we shall consider all the canonical variables on the same footing. Capital latin indices will range over $2n$ values. In equation (1) we have the canonical Poisson tensor

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{2}$$

in Darboux form and $[\ ,\ ]_0$ is the usual Poisson bracket. Assuming these equations of motion to be super-integrable, we shall have $2n - 2$ further first integrals of motion in addition to the usual Hamiltonian function H^1. Every maximally super-integrable system admits a set of first integrals H^α where Greek indices will be reserved to range over $2n - 1$ values.

Due to the functional independence of these first integrals, their gradients $\partial H^\alpha / \partial X^A$ define $2n - 1$ linearly independent vectors orthogonal to the velocity vector. By taking their full cross-product we determine at each point in phase space a unique direction which is precisely that of the velocity vector. Then the trajectory is determined by the equations of motion in the Nambu form

$$\begin{aligned} \dot{X}^A &= \frac{\partial(H^1, \ldots, H^{2n-1})}{\partial(X^1, \ldots \widetilde{X^A} \ldots X^{2n})} \\ &= \varepsilon_{\alpha_1 \ldots \alpha_{2n-1}} \epsilon^{A_1 \ldots (A) \ldots A_{2n}} \frac{\partial H^{\alpha_1}}{\partial X^{A_1}} \cdots \frac{\partial H^{\alpha_{2n-1}}}{\partial X^{A_{2n}}} \end{aligned} \tag{3}$$

where a tilde over a quantity indicates that it will be omitted and indices enclosed by round parentheses are excluded from the implied summation. Here $\epsilon^{A_1 \ldots A_{2n}}$ with

$$\epsilon^{123 \ldots 2n} = \frac{1}{\sqrt{g}} \tag{4}$$

is the $2n$-dimensional completely anti-symmetric Levi-Civita tensor density in phase space, whereas $\varepsilon_{\alpha_1 \ldots \alpha_{2n-1}}$ is the permutation symbol in $2n-1$ dimensions. The factor of proportionality $\sqrt{g}$ will be determined from the requirement that both the magnitude and the direction of the velocity vector in phase space are given by the original equations of motion. It will be recognized as the volume density in phase space. For the class of super-integrable systems that we shall consider here $\sqrt{g}$ will be time independent, that is, a function of integrals of motion H^α only. See [21] for cases where this fails and Nambu mechanics is generalized.

In Nambu's form of the equations of motion (3) it is immediately evident that they can be expressed in bracket form in $2n - 1$ different ways

$$\dot{X}^A = [X^A, H^\alpha]_\alpha = J_\alpha^{AB} \frac{\partial H^\alpha}{\partial X^B} \tag{5}$$

depending on which one of the conserved quantities H^α we take as the Hamiltonian function. In (5) there is no summation implied over α which simply enumerates the different ways the original equations of motion can be written in maximally degenerate Poisson bracket form. The labels of both the Hamiltonian function and the degenerate Poisson bracket must coincide. The expression for the $2n - 1$ maximally degenerate Poisson tensors will then be

$$J_\alpha^{AB} = \varepsilon_{\alpha_1 \ldots (\alpha) \ldots \alpha_{2n-1}} \epsilon^{A_1 \ldots (A) \ldots (B) \ldots A_{2n}} \frac{\partial H^{\alpha_1}}{\partial X^{A_1}} \cdots \frac{\partial \widetilde{H^\alpha}}{\partial X^B} \cdots \frac{\partial H^{\alpha_{2n-1}}}{\partial X^{A_{2n}}} \tag{6}$$

whereby (5) and (6) together will simply yield (3). This point follows from [17] (see also [21, 22]). Note that the equations of motion can be obtained in two different ways using the same Hamiltonian function H^1 with either Darboux's canonical Poisson tensor (2), or the first one of the degenerate Poisson tensors constructed according to (6) with $\alpha = 1$. The proof of the Jacobi identity

$$J_\alpha^{M[A} \frac{\partial J_\beta^{BC]}}{\partial X^M} = 0 \tag{7}$$

where square brackets denote complete skew-symmetrization, and the fact that all these Poisson tensors are degenerate

$$\det J_\alpha^{AB} = 0 \tag{8}$$

can be found in [17]. From the construction (6) it is manifest that each J_α admits $2n-2$ Casimirs H^β with $\alpha \neq \beta$. Thus, in all cases the rank of the Poisson tensors J_α^{AB} will be 2, independent of the dimension of the dynamical system. These maximally degenerate Poisson tensors are compatible and thus form a Poisson pencil, a fact manifest from the lack of any restriction we have put on α and β in the Jacobi identity (7). However, none of the degenerate Poisson tensors is compatible with the Poisson tensor (2) in Darboux's canonical form. The 'basic identity' discussed in the literature on Nambu brackets is simply the Jacobi identity (7) for maximally degenerate Poisson brackets.

We shall now present the maximally degenerate Poisson tensors for the harmonic oscillator in two dimensions, $n = 2$, as the simplest non-trivial example of this subject. The equations of motion are given by the usual Hamiltonian function

$$H^1 = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}k(x^2 + y^2) \tag{9}$$

in the form of Hamilton's equations with Darboux's Poisson tensor (2). But the harmonic oscillator is super-integrable, that is, it admits two further first integrals

$$H^2 = \frac{1}{2m}\left(p_x^2 - p_y^2\right) + \frac{1}{2}k(x^2 - y^2) \tag{10}$$

$$H^3 = \frac{1}{2}L^2 \qquad L \equiv xp_y - yp_x \tag{11}$$

where H^3 is the squared magnitude of angular momentum and H^2 is the super-integral which comes from the fact that the equations of motion decouple in Cartesian coordinates. This choice of first integrals is such that we can generalize our discussion [20] to completely integrable Smorodinsky–Winternitz potentials [19] of the Schrödinger equation in two dimensions. The volume density is given by

$$\sqrt{g} = 2LM \qquad M \equiv \frac{1}{m}p_x p_y + kxy = \sqrt{(H^1)^2 - (H^2)^2 - 2\omega^2 H^3} \tag{12}$$

which is manifestly a conserved quantity. Here $\omega^2 = k/m$ is the frequency of oscillation. Perhaps in (12) we should have used the symbol $(H^2)'$ instead of M because if we had started with M as the super integral, then our H^2 in (10) would enter into the volume factor. The three degenerate Poisson tensors that follow from the construction (6) are given by

$$\begin{aligned}
&[x, y]_1 = \frac{1}{2mM}(xp_x - yp_y) \qquad &&[p_x, p_y]_1 = -\frac{k}{2M}(xp_x - yp_y)\\
&[x, p_x]_1 = \frac{1}{2} &&[y, p_y]_1 = \frac{1}{2}\\
&[x, p_y]_1 = \frac{1}{2M}\left(\frac{1}{m}p_x^2 + ky^2\right) &&[y, p_x]_1 = \frac{1}{2M}\left(\frac{1}{m}p_y^2 + kx^2\right)
\end{aligned} \tag{13}$$

together with

$$\begin{aligned}
&[x, y]_2 = -\frac{1}{2mM}(xp_x + yp_y) \qquad &&[p_x, p_y]_2 = \frac{k}{2M}(xp_x + yp_y)\\
&[x, p_x]_2 = \frac{1}{2} &&[y, p_y]_2 = -\frac{1}{2}\\
&[x, p_y]_2 = -\frac{1}{2M}\left(\frac{1}{m}p_x^2 - ky^2\right) &&[y, p_x]_2 = \frac{1}{2M}\left(\frac{1}{m}p_y^2 - kx^2\right)
\end{aligned} \tag{14}$$

and the third one is the simplest

$$[x, y]_3 = -\frac{1}{m^2LM}p_xp_y \qquad [p_x, p_y]_3 = -\frac{k^2}{LM}xy$$
$$[x, p_x]_3 = 0 \qquad [y, p_y]_3 = 0 \tag{15}$$
$$[x, p_y]_3 = \frac{\omega^2}{LM}yp_x \qquad [y, p_x]_3 = -\frac{\omega^2}{LM}xp_y$$

and has an interesting free particle limit $[x, y]_3 = -1/mL$ with all others vanishing. This is reminiscent of the Dirac bracket [23] in the Landau problem for the motion of a charged particle in a strong magnetic field with angular momentum replacing the magnetic field. Curtright and Zachos [14] have discussed the relationship between Dirac and Nambu brackets.

We see that the maximally degenerate Poisson tensors constructed according to (6) are highly nonlinear, which will pose nasty problems if we were to carry them over to the quantum-mechanical domain naively. However, there is a way to overcome this difficulty. By introducing new coordinates in phase space that consist of $\{H^\alpha\}$ and H^{2n} which is a 'time' variable

$$\frac{\mathrm{d}H^{2n}}{\mathrm{d}t} = 1 \tag{16}$$

the maximally degenerate Poisson bracket relations can be summed up in the form

$$[H^\alpha, H^\beta]_\gamma = 0 \qquad \forall\,\alpha, \beta, \gamma \tag{17}$$

$$[H^{2n}, H^\alpha]_\beta = \delta^\alpha_\beta \tag{18}$$

of Heisenberg's equations.

For the harmonic oscillator the choice

$$H^4 = \frac{1}{\omega}\tan^{-1}\left(m\omega\frac{x+y}{p_x+p_y}\right) \tag{19}$$

satisfies (16). We note that this choice of H^4 is not unique, a point which will emerge as immaterial in what follows because we shall only need its gradients and they are directly obtained from the equations of motion. It requires the usual straightforward but lengthy calculations to verify the properties (17), (18) in the case of the harmonic oscillator.

Another advantage of introducing privileged coordinates in phase space is that now a Riemannian metric on phase space is suggested. In the example of the harmonic oscillator this is given by

$$\mathrm{d}s^2 = 2H^1\,\mathrm{d}H^1\,\mathrm{d}H^3 + 2H^2\,\mathrm{d}H^2\,\mathrm{d}H^3 + 2\omega^2\,\mathrm{d}(H^3)^2 + H^3[\mathrm{d}(H^1)^2 - \mathrm{d}(H^2)^2] + \mathrm{d}(H^4)^2 \tag{20}$$

which has the determinant given by (12).

3. Quantum operators for maximally degenerate Poisson brackets

In elementary quantum mechanics courses we are taught that the passage to quantum mechanics requires

$$[p_i, q^k]_0 = \delta^k_i \quad \longrightarrow \quad \hat{p}_i\hat{q}^k - \hat{q}^k\hat{p}_i = \frac{h}{\mathrm{i}}\delta^k_i \tag{21}$$

where hats denote operators, and are shown Schrödinger's solution to it

$$\hat{p}_i = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q^i} \qquad \hat{q}^k = q^k \tag{22}$$

which always mystifies. Now we shall propose a definition which makes sense out of this.

Definition. *Given a Poisson tensor J and* any *function of the canonical variables F, the quantum operator $\hat{F}$ that corresponds to F is given by*

$$\hat{F} = \frac{\hbar}{\mathrm{i}} J^{AB} \frac{\partial F}{\partial X^A} \frac{\partial}{\partial X^B} \tag{23}$$

which is proportional to the Poisson vector field appropriate to F.

With this definition we can immediately see that Schrödinger's solution (22) is obtained from Darboux's Poisson tensor (2) and the functions p_i. Of course, we do not apply this definition to the functions q^i but instead take them to be c-numbers, because then $\hat{p}_i$ and $\hat{q}^i$ would commute contradicting (21).

This is not the appropriate place to discuss the merits of definition (23), or any lack thereof. We shall instead apply it to the degenerate Poisson brackets and see if we arrive at a consistent theory.

From equations (17) and (18), it becomes clear that in the case of quantization with degenerate Poisson brackets we need to regard each H^α as a c-number and construct the quantum operators using H^{2n}. There will be $2n-1$ such operators

$$\hat{H}^{2n}_\alpha = \frac{\hbar}{\mathrm{i}} J^{AB}_\alpha \frac{\partial H^{2n}}{\partial x^A} \frac{\partial}{\partial x^B} \tag{24}$$

corresponding to the full set of degenerate Poisson brackets. All of these operators must commute

$$\left[\hat{H}^{2n}_\alpha, \hat{H}^{2n}_\beta\right] = 0 \qquad \forall\, \alpha, \beta \tag{25}$$

or more precisely, their Lie brackets must vanish. These are very strong requirements, but they will be satisfied because we are dealing with maximally super-integrable systems.

Let us illustrate this with the example of the quantum operators for the harmonic oscillator. From definition (24) using the degenerate Poisson brackets (13)–(15) with H^4 given by (19), we arrive at the following quantum operators:

$$\hat{H}^4_1 = \frac{\hbar}{2\mathrm{i}LM}\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + p_y\frac{\partial}{\partial p_x} + p_x\frac{\partial}{\partial p_y}\right) \tag{26}$$

$$\begin{aligned}\hat{H}^4_2 = \frac{\hbar}{4\mathrm{i}M(H^1+M)}\bigg\{&-\frac{1}{m}\left(p_x^2 - p_y^2\right)\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + k(x+y)^2\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\\ &-k(x^2-y^2)\left(p_y\frac{\partial}{\partial p_x} + p_x\frac{\partial}{\partial p_y}\right) + \frac{2}{m}p_x p_y(x+y)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\\ &+2kxy(p_x+p_y)\left(\frac{\partial}{\partial p_x} - \frac{\partial}{\partial p_y}\right) + \frac{1}{m}(p_x+p_y)^2\left(p_y\frac{\partial}{\partial p_x} - p_x\frac{\partial}{\partial p_y}\right)\bigg\}\end{aligned} \tag{27}$$

and

$$\begin{aligned}\hat{H}^4_3 = \frac{\hbar}{2\mathrm{i}LM(H^1+M)}\bigg\{&\left[ky(x+y) + \frac{1}{m}p_y(p_x+p_y)\right]\left(\frac{1}{m}p_x\frac{\partial}{\partial x} - kx\frac{\partial}{\partial p_x}\right)\\ &-\left[kx(x+y) + \frac{1}{m}p_x(p_x+p_y)\right]\left(\frac{1}{m}p_y\frac{\partial}{\partial y} - ky\frac{\partial}{\partial p_y}\right)\bigg\}.\end{aligned} \tag{28}$$

It is a straightforward but most lengthy calculation to verify that all Lie brackets of (26), (27) and (28) indeed do vanish.

We note that earlier Hietarinta [10] had discussed Nambu tensors and commuting vector fields. Hietarinta's vector fields are just the Hamiltonian vector fields obtained from Darboux's Poisson tensor (2). It is interesting to compare the operators (26)–(28) with Hietarinta's Hamiltonian vector fields for the harmonic oscillator

$$H^1 = -\frac{1}{m} p_x \frac{\partial}{\partial x} + kx \frac{\partial}{\partial p_x} - \frac{1}{m} p_y \frac{\partial}{\partial y} + ky \frac{\partial}{\partial p_y} \tag{29}$$

$$H^2 = -\frac{1}{m} p_x \frac{\partial}{\partial x} + kx \frac{\partial}{\partial p_x} + \frac{1}{m} p_y \frac{\partial}{\partial y} - ky \frac{\partial}{\partial p_y} \tag{30}$$

$$H^3 = L \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + p_y \frac{\partial}{\partial p_x} - p_x \frac{\partial}{\partial p_y} \right) \tag{31}$$

which consists of the generators of three commuting rotations in four dimensions. Finally, we note that the operators $\hat{H}^{\alpha}_{\alpha}$ obtained from Nambu–Poisson brackets in accordance with the definition (23) reduce to $\hat{H}^1_1 = \hat{H}^2_2 = \hat{H}^3_3 = H^1$ of (29).

We have seen that the maximally degenerate Poisson quantum operators assume formidable expressions even in the case of the simplest possible classical system. To quantize we need to set up $2n - 1$ eigenvalue problems

$$\hat{H}^{2n}_{\alpha} \Phi = \lambda_{\alpha} \Phi \tag{32}$$

for some state vector Φ which is *not* the Schrödinger wavefunction. This looks like a very difficult problem, but its solution is simplicity itself,

$$\Phi = \exp \frac{\mathrm{i}}{\hbar} \sum_{\alpha=1}^{2n-1} \lambda_{\alpha} H^{\alpha} \tag{33}$$

where we recall that H^{α} are the privileged coordinates in phase space[1]. But these are also conserved quantities and therefore

$$\dot{\Phi} = 0 \tag{34}$$

and we have a frozen time formalism. In particular, from (32) and the expression for the eigenfunction (33) we find that for the case of the harmonic oscillator λ_1 is the energy, λ_3 is the magnitude of the angular momentum and λ_2 is the constant of separation in the Schrödinger equation.

The quantization condition is simply the requirement of single-valuedness of this eigenfunction. That is, Φ must be periodic in the privileged coordinates H^{α}. This forces the eigenvalues λ_{α} to become integers. The specific nature of these eigenvalues will be determined by the holonomy structure of Φ which must be done on a case-by-case basis.

[1] This result has a counterpart in usual quantum mechanics. For the Schrödinger wavefunction, the solution of the eigenvalue problem

$$\hat{p}_i \Psi = p_i \Psi$$

is the plane wave

$$\Psi = \exp \frac{\mathrm{i}}{\hbar} \sum_{i=1}^{n} p_i q^i$$

quite analogous to (32) and (33). This parallel shows the virtue of our definition (23) for turning classical functions into quantum operators. There is, however, no resemblance between the state vectors Φ and Ψ in terms of their physical meaning.

4. Conclusion

Developments in the theory of Poisson structure have not yet made their full impact on quantum mechanics as they certainly will. In this paper we have considered only one but an important aspect of these developments, namely the maximal set of degenerate Poisson brackets derived from Nambu's form of the equations of motion for super-integrable systems. We have shown that these Poisson brackets define skew-symmetric tensors of second rank satisfying the Jacobi identity. The highly nonlinear expressions we find for these brackets become Heisenberg equations when we introduce new coordinates in phase space that consist of the first integrals of motion and a 'time' variable. We have proposed a definition for turning classical functions into quantum operators and with its help formulated a set of eigenvalue problems for super-integrable systems. We emphasize again that the eigenfunction we have introduced is an entirely new object that has nothing to do with Schrödinger's wavefunction. The solution for the eigenfunction results in a phase factor which consists of a linear superposition of the privileged coordinates on phase space. The requirement of its single-valuedness is the quantization condition. The determination of the precise nature of the eigenvalues is based on the holonomy structure of Φ.

This is a general procedure for alternative quantization of maximally super-integrable systems. However, it is not the full story because even in the dynamical systems with three degrees of freedom that Nambu first discussed, there are cases where the factor of proportionality $\sqrt{g}$ is not conserved and Nambu mechanics must be generalized [21].

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